Chaos based on Riemannian geometric approach to SU(2) Yang-Mills classical field theory(5) Quantum chaos in field theory and cosmology, Chaos and Nonlinear Dynamics in Quantum-Mechanical and Macroscopic Systems)

AUTHOR(S):

Kawabe, Tetsuji; Koyanagi, Shinichiro

Chaos based on Riemannian geometric approach to SU(2) Yang-Mills classical field theory

Phys. Dept, Dept. of Acoustic Design, Kyushu University Tetsuji Kawabe [1] and Shinichiro Koyanagi

ハミルトン系カオスをリーマン多様体の曲率の変動に関係づけて理解しようという研究がなされている．このリーマン幾何学的アプローチをヤンミルズ場方程式に適用して，ウー・ヤン・モノポール解のリヤプノフ指数を計算した．従来のエルゴード理論に基づく結果と比較することができるので，この幾何学的アプローチの有効性や妥当性なども議論できる．

Abstract

Using the Riemannian geometric approach to the Hamiltonian dynamics, we have studied the chaos in the Wu-Yang monopole solution of the SU(2) Yang–Mills field equation. It is shown that the features of chaos can be characterized by such geometric quantities as the average and the fluctuation of the Ricci curvature of the mechanical manifold.

1 Riemannian Geometric Approach to the SU(2) YM equation

Basic idea of this approach starts with a picture that the instability of the trajectories of a dynamical system can be viewed as the instability of geodesics on a Riemannian manifold endowed with a suitable metric [1,2]. The instability of the geodesic is characterized by the Jacobi field $\boldsymbol{J}$ which measures the spread between nearby geodesics. The evolution for the norm of the Jacobi field is described by a stochastic oscillator equation

$$\frac{d^2||\boldsymbol{J}||}{ds^2} + k(s)||\boldsymbol{J}|| = 0 \ , \tag{1}$$

whose solution gives the maximal Lyapunov exponent λ as

$$\lambda(k_0, \sigma_k) = \frac{1}{2}\left(\Lambda - \frac{4k_0}{3\Lambda}\right) \ , \tag{2}$$

where

$$\Lambda = \left(\sigma_k^2\tau + \sqrt{\left(\frac{4k_0}{3}\right)^3 + \sigma_k^4\tau^2}\right)^{\frac{1}{3}}, \qquad \tau = \frac{\pi\sqrt{k_0}}{2\sqrt{k_0(k_0+\sigma_k)} + \pi\sigma_k} \ . \tag{3}$$

Here k_0 is the mean of the Ricci curvature K_R per degree of freedom, i.e., $\langle K_R \rangle/(N-1)$, and σ_k^2 is its variance. The Ricci curvature is given by $K_R = \Delta V$ for the Eisenhard metric.

The SU(2) YM equation we consider is

$$(\partial_t^2 - \partial_r^2)\phi(r,t) = \frac{1}{r^2}\phi(r,t)[1-\phi(r,t)^2] \ . \tag{4}$$

[1] E-mail: kawabe@design.kyushu-u.ac.jp

The discretized Hamiltonian of (4) is given by

$$H = \sum_{i=1}^{N} \frac{1}{2}\left(\frac{\partial \phi_i}{\partial t}\right)^2 + V(\phi_i) , \tag{5}$$

$$V(\phi_i) = \sum_{i=1}^{N+1} \frac{1}{2h^2}(\phi_i - \phi_{i-1})^2 + \sum_{i=1}^{N} \frac{1}{4(ih)^2}(1-\phi_i^2)^2 , \tag{6}$$

where the coordinate r is subdivided into a small interval of length h as $r_i = ih$, with $i = 1, 2, \ldots, N$, and the function ϕ_i stands for the gauge field $\phi(r_i, t)$. The time average of the Ricci curvature is then calculated by

$$\langle K_R \rangle = \langle \Delta V \rangle = \frac{1}{T}\int_t^{t+T} K_R \, dt = \frac{1}{h^2}\left[2 + \frac{1}{N}\sum_{i=1}^{N}\int_t^{t+T} \frac{1}{i^2}(3\phi_i^2 - 1)dt\right] \tag{7}$$

2 Results

For the Wu–Yang monopole solution $\phi(r) = 0$, we have calculated k_0, σ_k, and λ as a function of the energy density $\varepsilon = E/N$, i.e., the energy per degree of freedom.

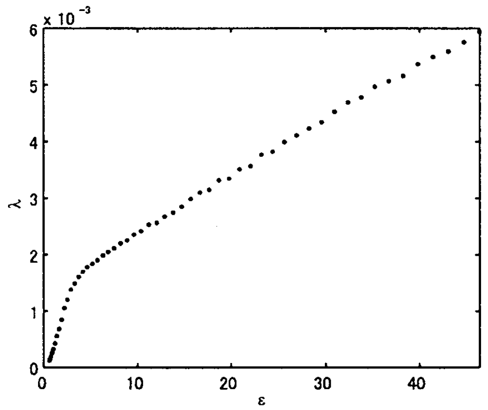

Fig. 1: λ vs ε

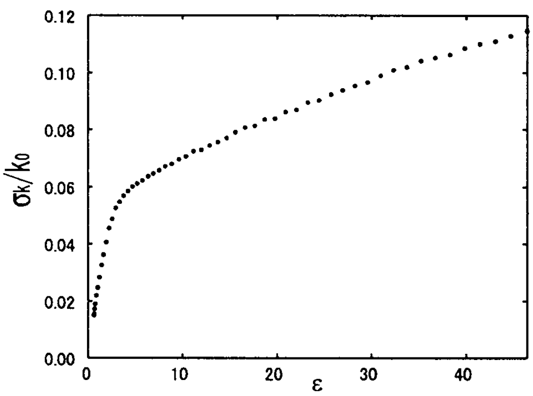

Fig. 2: σ_k/k_0 vs ε

Fig.1 shows the dependence of λ on ε. This feature seems to be consistent with the result obtained by the conventional approach [3]. Fig.2 shows the dependence of the ratio σ_k/k_0 on ε, which suggests that the weakly chaotic regime corresponds to small curvature fluctuation regime ($\varepsilon \to 0$) while the strongly chaotic regime to large fluctuation regime ($\varepsilon \gg 0$). That is, the degree of chaoticity is characterized by σ_k/k_0.

References

1) L.Casetti, M.Pettini, and E.Cohen, Phys. Rep. **337** (2000), 237

2) T.Kawabe, Phys. Rev. E **67** (2003), 016201

3) T.Kawabe, S.Ohta, Phys. Rev. D **41** (1990), 1983